A Fresh Spin on Newton's Bucket

Robert L. Shuler, Jr., NASA Johnson Space Center, Houston, TX

July 7, 2014 (rev. Dec. 2014)

ABSTRACT: A simple mathematical formulation of Mach's principle is given based on a century of investigation into inertia, and used to check the results of Newton's famous bucket experiment.

An introduction to physics seems incomplete without the thought experiment known as Newton's Bucket. Doing so also introduces the famous historical critique of Newton by Mach, which inspired Einstein and spawned modern theories of gravity and the cosmos. The critique often fascinates and inspires students. But Mach's concept is not expressed in a simple algebraic way, which the students can analyze as easily as Newton's concepts. That gap can be filled using ideas about inertia from Einstein, Sciama, Ciufolini and Wheeler, and others over the last century. It gives results which may have surprised Mach, certainly will teach students how to think clearly about relative motion, and provides an introduction to inertia theory with the context of General Relativity.

The disagreement of Mach with Newton concerns whether motion is defined entirely relative to surrounding bodies. Newton thought that it was not, and maintained that if a bucket of water were placed in an otherwise empty universe, and set spinning, that eventually the water in the bucket would begin to spin also, and would rise up the side of the bucket forming a concave shape. This argument appears at the beginning of Newton's *Principia*, published in 1687, introducing his famous laws of motion and his theory of gravity. Both remain so accurate that they are all that is needed for the calculation of precision trajectories by which spacecraft visit other planets.

Writing 200 years later in his *The Science of Mechanics*, Ernst Mach devoted a section to a philosophical rebuttal of Newton. To Mach – the same Mach who gave us "Mach numbers" for describing supersonic flight – motion did not make any sense except in relation to some other body. Newton's concept of absolute space seemed to irritate him, and this section of his book is something of a diatribe which he expanded in subsequent editions. Mach's book was in use for a long time, lasting through at least 7 editions, and his views impressed Einstein. It was Einstein who coined the term "Mach's principle." According to Mach, not only would the water in the bucket experience no tendency to escape the bucket in an otherwise empty universe, but if you could somehow spin our universe about the bucket as in Figure 1, then the surface of the water would become concave, just the same as if the bucket were spinning. Only relative motion between the bucket and the universe could be important.

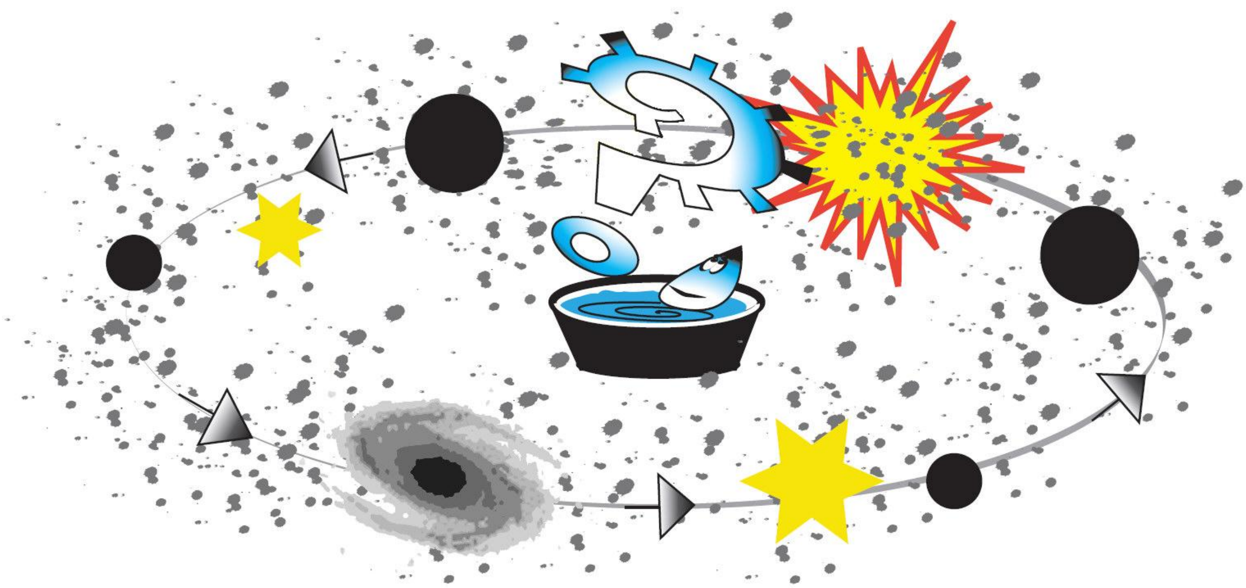

Fig. 1: A drop in Mach's bucket ponders which way it should go

illustration by Crystal Wolfe

Fascination with Mach's thought experiment has not only infected students, but practically every mass or gravity investigator since then. For one thing, it is subject to testing – not because we can actually spin the universe (though we might find it to be spinning, it doesn't seem to be), but because Mach did *not* leave a detailed mathematical formulation as Newton did. So we can make up formulations based on other theories, and test to see what kind of explanation of Mach's experiment they might give. But for another thing, it has an astounding implication – because the water does not fly out of the bucket in an empty universe, Mach seems to be saying that the *inertia* of the water, it's kinetic mass, is due entirely to the presence of other masses in the universe. When Mach’s principle is discussed in scientific literature, this is often the aspect which is of interest.

Einstein was certainly hooked, and gave a Machian formulation of mass in 1912 [1], several years before his theory of General Relativity. He supposed that inertial mass was related to gravitational potential energy. The total mass of an object would be the negative (since gravitational potential is negative, we reverse it to get a positive mass) of the sum of gravitational potential between an object and every other object in the universe. This intuitive idea can explain how inertial and gravitational mass are always proportional to one another, a part of the Equivalence Principle, but it gives only an equation for scalar inertia. It still relies on Newton’s laws of motion. In fact it does not help at all with the problem of the bucket, for a reason which Einstein would eventually discover. Einstein thought he incorporated Mach’s principle into General Relativity (GR), and defended this position for decades. But there was a problem. GR allows an empty universe, and a small test particle in such a universe still has inertia.

In 1953 Dennis Sciama continued this line of thought, formulating inertia as a back-reaction to acceleration in the gravitational fields of remote objects, analogous to the back-EMF of electric coils (inductors) which resist changes in current flow [2]. This retained Einstein’s relation of inertia to gravitational potential, and added a vector dependence on relative acceleration. Sciama promised a second paper linking the work to GR, but instead his colleague Davidson concluded that the effect was already contained within GR [3]. Later proponents such as Ciufolini and Wheeler have suggested only a qualitative relationship between gravitational potential and inertial mass [4].

The analogy to electromagnetic effects led some to ask if inertia could be anisotropic, that is, if inertia would appear only for acceleration toward or away from nearby inertia-inducing masses, such as galaxies. Experiments were devised and no such anisotropy was found [5]. The physics community was for many years split along these lines, with advocates of Machian theories (e.g. Brans and Dicke) arguing that induced inertia would be isotropic, and advocates of quantum particle theories of inertia (e.g. Weinberg) arguing that Machian theories should be eliminated because no anisotropy was found.

Then an argument was given based on the Equivalence principle, that Machian inertia should be isotropic [6]. Inertia was linked to time dilation. Under time dilation, inertia is increased by the same amount as time is dilated. This is similar to the case with Special Relativity (SR), where externally observed mass increase and time dilation occur by the same factor $\gamma = 1/\sqrt{1-(v^2/c^2)}$. Objects or processes lower in a gravitational field (or in an accelerated reference frame) also experience a well known time dilation predicted by General Relativity. Time dilation has been extensively observed, and it is necessary to compensate for it in order for GPS satellites and systems to function properly. An object in a time dilated reference frame also appears, to the higher observer, to have an increased mass, by the same amount as the time dilation. In an exaggerated case of this, objects which approach the event horizon of a black hole, even light, slow down and approach zero velocity at the horizon. And it is very hard to extract them (evidence of their increased inertia). It is very hard to move them in any direction. The inertia is isotropic.

Trajectories of objects falling into a black hole are normally described in the reference frame of the falling object, using time as experienced by the object – which is called “proper time.” In proper time, the object notices no change in its own inertia. As its response to force slows, so does its clock, indeed all its perceptions. This is well known. If we take this idea in the other direction, and apply it to an object which is removed far away from other masses, so that in the limit it approaches the case of a test particle in an empty universe, we see that all its perceptual processes speed up at the same rate as its inertia, and thus test particles in an empty universe still have inertia in their own proper time, which is the only time reference available. So perhaps inertia in an empty universe is not entirely to be unexpected, even in a Machian formulation, if one uses a mathematical limit approach.

Unless one has a separate way of measuring “gravitational charge” (we don’t), then all the Machian theories have a circularity. They define inertial mass in terms of gravitational mass, which as far as we can measure are the same thing. One might hope for a theory which explains the origin of both inertial and gravitational mass. But meanwhile, we can use the ordinary mass we measure as a “proxy” for a hypothetical mass-causing property of matter and energy particles. We’ll call this proxy m_i, whereby we mean the mass-causing potential of the i^{th} object or particle or galaxy etc., and the number we use is what we measure as its ordinary rest mass.

This is simply a weighting factor which, together with distance R_i, determines the relative influence of other objects on a test particle of interest m_0, where R_i is the distance between the objects. The reciprocal of distance between objects (*1/R*) determines gravitational potential. This is true of all of the inertia theories cited above, even though they are reasoned from different bases, from Special Relativity in the case of Einstein 1912, and radiation reaction in the case of Sciama 1953, or broadly and approximately from potential energy in the case of Ciufolini and Wheeler. In all cases the contribution to the inertia of a test particle m_0 from an object m_i is approximately the potential energy contribution of m_i to m_0 (or rather the negative of potential energy, since potential is by convention negative) (G is the gravitational constant):

$$m_{inertia\_contribution\_to\_m_0\_from\_m_i} = Gm_0 m_i / R_i c^2 \tag{1}$$

Summing over all the masses *i* in the universe (or at least over all those not outside the cosmic light horizon, according to Sciama), the inertia theories conclude the mass of an object m_0 is governed by a relationship to other masses like this:

$$\begin{aligned} & m_0 \approx \sum_i Gm_0 m_i / R_i c^2 \\ & \Leftrightarrow 1 \approx \sum_i Gm_i / R_i c^2 \end{aligned} \tag{2}$$

Sciama used this relation in the early 1950s to predict that a great deal more mass would be found by astronomical observations, and Ghosh reported in 2000 that approximately the right amount of mass had been found [7]. We may examine how inertia would be increased by proximity to a particularly nearby and massive object of interest *j* by taking it out of the summation like this:

$$\begin{aligned} & m_0' \approx m_0 [\sum_{i \neq j} Gm_i / R_i c^2 + Gm_j / R_j c^2] \\ & \Rightarrow m_0' \approx m_0 (\sim 1 + Gm_j / R_j c^2) \end{aligned} \tag{3}$$

As noted there are some approximations in this formula. We do not expect it to be exact. But it is interesting that it closely matches the time dilation formula of the Schwarzschild metric, a typical measure of the effects of proximity to large objects in General Relativity. As noted above, in inertia theory, time dilation must match inertia increase. Using the approximations that for $x<<1$ we have $1/(1-x)\approx 1+x$ and $(1-x)^n \approx 1-nx$ (Einstein often used such approximations) the Schwarzschild time dilation can be written to look like (3):

$$\begin{aligned}
&\Delta t_0 = \Delta t_0{}'(1-2Gm_j / R_j c^2)^{\frac{1}{2}} \\
&\Leftrightarrow \Delta t_0{}' = \Delta t_0 / (1-2Gm_j / R_j c^2)^{\frac{1}{2}} \\
&\Rightarrow \Delta t_0{}' \approx \Delta t_0 / (1-Gm_j / R_j c^2) \\
&\Rightarrow \Delta t_0{}' \approx \Delta t_0 (1+Gm_j / R_j c^2)
\end{aligned} \tag{4}$$

For radii beyond about 10 times the Schwarzschild radius $R_0 = 2Gm/c^2$, differences between (4) and (3) become essentially insignificant, so for our purposes we can use (3) and it makes the algebra a bit simpler. The Schwarzschild radius of one solar mass is only about 2 miles, so nothing in the solar system is dense enough to invalidate this approximation.

A further demonstration of consistency between General Relativity and inertia theory is possible by using the Hamiltonian formulation that the total energy of a system *H* is the sum of kinetic energy *T* and potential energy *V*. This analysis can be found in [8].

To analyze the bucket quantitatively, we can postulate an equation like Newton’s *F=ma* which is entirely relative in the Machian sense. It will be a summation, since there is a different relative acceleration $\mathbf{a}_i$ between the object of interest m_0 and all other objects m_i. Each acceleration will be weighted by the contribution $Gm_i / R_i c^2$ that m_i makes to the inertia of m_0:

$$\mathbf{F}_0 = \sum_i m_0 (Gm_i / R_i c^2)\mathbf{a}_i \tag{5}$$

Since the $\mathbf{a}_i$ are *relative* accelerations, i.e. the difference between the accelerations of the two objects measured in a particular frame, then they are *frame independent*. R_i’s are the distances between the objects (which are scalars since inertia is isotropic). And *c* is the velocity of light. The vector force $\mathbf{F}_0$ arises only if all of the relative weighted accelerations do not add to zero. If no force is applied then this will be the case. There is no notion of absolute acceleration. If all of the objects in the universe moved (accelerated) together, *no one would notice or feel any force!*

Now we are in a position to analyze definitely what would happen with Mach's version of the bucket, based on an equation that most would agree captures the essence of Mach's principle, and nothing more. If the m_i are taken to be "fixed stars" with respect to which an object has a single acceleration vector $\mathbf{a}$, which can be factored out of the summation, and the inertial mass m of the object m_0 is taken to be proportional to its gravitational potential $\sum_i Gm_0 m_i / R_i c^2$ as Einstein and Sciama suspected, then in fact (5) reduces to Newton's equation.

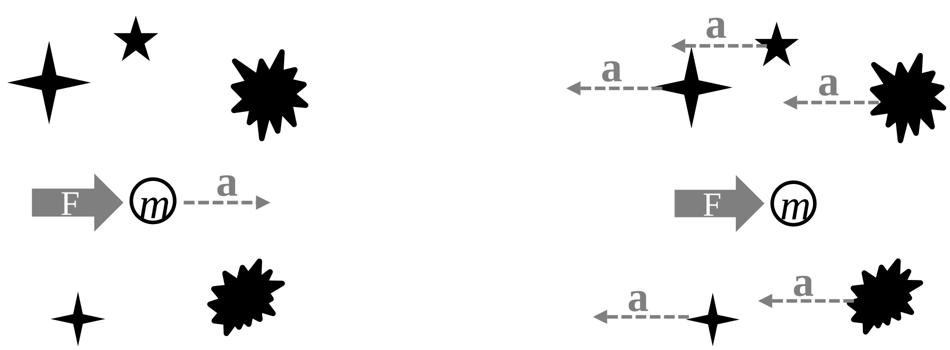

Fig. 2 (a): particle *m* accelerates (b): universe accelerates

Figures 2 (a) and 2 (b) show two views of the linear acceleration of a particle *m*. In (a) is the usual view, the "inertial" frame of the "fixed stars," and in (b) the frame of the particle. The same equation describes either, since only relative accelerations are used. Part (b) could be stated as a linear version of Mach's thought experiment: What if the universe accelerated? The answer is obvious. It would drag the particle with it, *unless* a force **F** were used to prevent the particle's acceleration. Qualitatively what (5) says is that relative positions and velocities remain the same unless some force is applied to change them. And it gives a weighting function so that in case parts of the universe accelerate in different directions, we can determine the net drag.

Note that we don't care how the various objects are moving in uniform motion. It is only changes in the relative motion, i.e. acceleration, that is noticed by equation (5), along with relative position which determines inertia (though of course relative uniform motion might eventually change inertia by changing distances). This exercise shows that our model behaves as expected, in a Machian manner. Now we will apply it to the bucket.

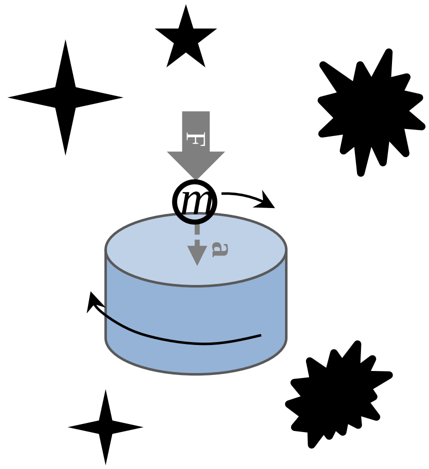

Fig. 3 (a): bucket rotating

Figure 3(a) shows a bucket rotating clockwise and analyzes a drop of water *m* on the edge of the bucket, using the reference frame of the surrounding stars and galaxies. To maintain its circular path, the drop *m* must be accelerated toward the center of the bucket. One infers that a force **F** must be supplied in this direction, and in a real bucket this is supplied by the pressure of water drops at the edge. In a gravitational field such as on Earth the water is piled up higher at the side of the bucket (the concave water surface). In space the spinning Earth, covered 3/4 with gravitationally bound oceans, finds them 21 km higher at the equator than at the poles.

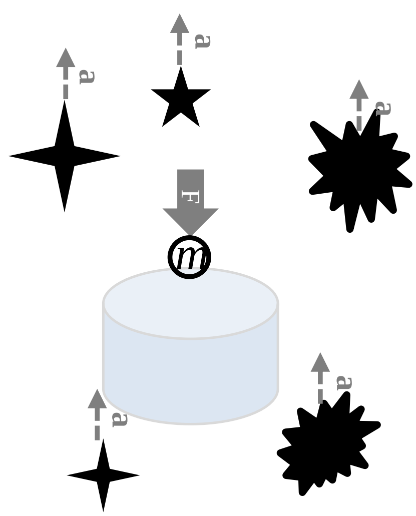

Fig. 3 (b): view from the reference frame of droplet *m*

Figure 3(b) shows the view from the reference frame of a single drop of water, labeled *m*. The droplet experiences a centripetal force, and sees the rest of the universe accelerate in the opposite direction. At each moment in time the direction changes, but the apparent acceleration of the rest of the universe is always aligned in the opposite direction to the centripetal force on *m*.

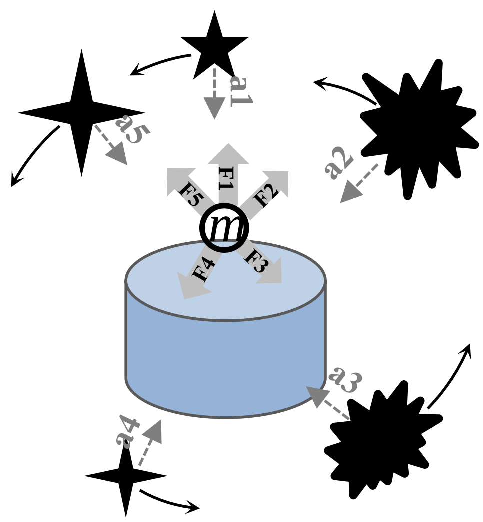

Fig 4: universe rotating

Figure 4 pictures the reference frame of a droplet in a stationary bucket as Mach described it. In this case, the apparent motion of the universe is the same for all droplets, given by Mach's specification that the universe is rotating about the bucket. Each cosmic object is seen to have an acceleration vector pointing toward the center of its circular orbit about the bucket, so that the net relative motion is the same as if the bucket were rotating. Indeed, Mach conjectured you wouldn't be able to tell the difference. However, in our *Machian* formulation of inertia (5), the acceleration of each cosmic object in an apparently *different* direction produces a plethora of small dragging impulses on the droplet *m* which instead of summing to a tendency for the droplet to fly off on a tangent, sum almost to zero. The drag is not quite zero, because the droplet is off center, but it is several orders of magnitude less than, and in the opposite direction from, what is expected! The corresponding net force needed to hold the droplet in position in the bucket is almost zero as well.

Thus if Mach is right, water in a bucket in a rotating universe nevertheless behaves as Newton claimed.

This finding is only one clue that rotational motion is not symmetric between two objects (or collections of objects) as is translational motion. If the bucket is rotating, the translational motion between the bucket and the astronomical objects is not affected and there is no time dilation or length contraction or synchronicity change due to the rotation. But if the galactic objects are revolving around the bucket, some of them would be going near the speed of light and would experience additional redshift due to time dilation, and some would have to go faster than light. If none of these effects were detected, then could we definitely say the universe was rotating? At the very least we can say that while the symmetry of translational motion is obvious, the symmetry of rotation is not.

REFERENCES

[1] A. Einstein, *The Collected Papers of Albert Einstein, Volume 4: The Swiss Years; Writings, 1912-1914. (English translation supplement)*, Translated by Anna Beck, Don Howard Consultant, ISBN 9780691026107 (1996, Princeton University Press, Princeton, NJ).

[2] D. W. Sciama, “On the Origin of Inertia,” Mon Not R Astron Soc, **113** (1953) pp. 34-42.

[3] W. Davidson, “General Relativity and Mach’s Principle,” Mon Not R Astron Soc, **117** (1957) pp. 212-224.

[4] I. Ciufolini and J. Wheeler, *Gravitation and Inertia* (Princeton University Press, Princeton NJ, 1995).

[5] G. Cocconi and E. E. Salpeter, Phys. Rev. Letters, “Upper Limit for the Anisotropy of Inertia from the Mossbauer Effect,” **4**, 4 (1960) pp.176-177.

[6] R. Shuler, “Isotropy, equivalence and the laws of inertia,” Phys Essays, **24**, 4 (2011) pp. 498-507.

[7] A. Ghosh, *Origin of Inertia – Extended Mach’s Principle and Cosmological Consequences* (Aperion, Montreal, Quebec Canada, 2000).

[8] Misner, C. W., Thorne, K. S., Wheeler, J. A., Gravitation, ISBN 0-7167-0344-0, W. H. Freeman & Co., New York, 1973.

AUTHOR INFORMATION:

Robert Shuler has been researching theories of inertia since 2007. He holds half a dozen patents, and has published in fields ranging from radiation tolerance in CMOS, to economics, to physics. Robert is the author of a book on the Equity Premium Puzzle, works for the Johnson Space Center and lives in Texas with his wife Natasha.