Leading Clocks Lag

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Abstract

A new memory aid for teaching the relativity of simultaneity is given that puts it on a par with "time dilation" and "length contraction" for quick and easy problem visualization. Guidelines for applicability of all three memory aids are given. As an example application the de Broglie wavelength is derived and dependence of de Broglie frequency on velocity explained in terms of Einstein synchronized reference frame measurements of a single clock (*2 on 1*) vs. measurements of an Einstein reference frame by an observer with a single clock (*1 on 2*).

Keywords: length contraction; simultaneity; special relativity; time dilation; de Broglie wavelength;

1. Introduction

The phrases "time dilation" and "length contraction" are easily remembered and convey an immediate sense of the character of observations involving relative motion. Einstein discusses in section 4 of his original paper on Special Relativity the "Physical Meaning of the Equations Obtained in Respect to Moving Rigid Bodies and Moving Clocks [1]," and even before that the term "contraction" was used by Lorentz [2]. But the third essential aspect of the Lorentz transform has no such shorthand expression. In this paper, for referring to the skew of clocks in another reference frame with respect to its direction of relative motion, the phrase *leading clocks lag* is introduced. While this is apparent from the negative sign on the x term of the Lorentz transform, recall that it also depends on velocity sign conventions and the reverse transform therefore switches sign, so when students are learning many new things it is easy to become confused.

In studies of student problem solving, as for example a recent report by Cormier and Steinberg [3], it is often found that students are most adept at visualizing and calculating with time dilation from the point of view of an inertial frame of reference. Length contraction is likewise an easy concept when applied to material objects, but causes confusion when no material object occupies a gap of space. Conflicting views of *simultaneity* in different reference frames even more frequently cause confusion. Is this because changing simultaneity is inherently more complex than slow running clocks or changes in length?

Clocks on the earth are not simultaneous. Most people have no difficulty with time zones, and the relativity of time to the position of an observer on the earth. There is nothing particularly easy about clocks that run slow or measuring rods that change length, yet these are more easily conceptualized by students than changes in simultaneity. And so the author suggests that perhaps an unequal job has been done of presenting the concept of relativistic simultaneity. The Lorentz transform is a coordinate transform which takes in one sort of coordinate point and gives out another. The phrases "time dilation" and "length contraction" refer instead to *intervals between pairs of coordinate points*, in the one case the readings of a clock at a fixed point in a coordinate system K at two different times, the endpoints of a "time ruler," and in the other case length of a spatial ruler extending between two coordinate points.

Simultaneity combines the two ideas, and compares the readings of two clocks separated by a spatial interval, at what appears in coordinate system K to be a single moment of time. Therefore by definition in K they have the same reading. But in system K' moving relatively to K at constant velocity v, all of the facts of clock rates, lengths, and simultaneity change.

2. Derivation of "leading clocks lag"

Consider relatively moving coordinate systems K and K' as shown in Figure 1.

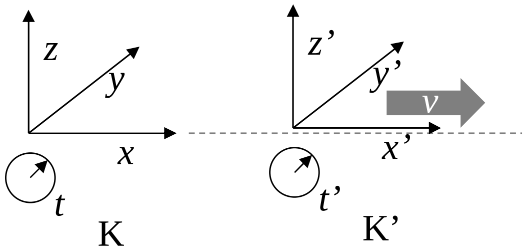

Fig. 1 – Relatively moving coordinate systems K and K’

The Lorentz transform for changing coordinate points from K to K’ is well known, and assuming we take the $+x$ axis to be the direction of motion of K in K’ as shown in Figure 1, it can be written:

$$\begin{aligned} y' &= y \\ z' &= z \\ x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx/c^2) \\ \gamma &= 1/\sqrt{(1 - v^2/c^2)} \end{aligned} \qquad (1)$$

Note that In system K the relative velocity of K’ is –v, so minus signs in x and t transformations become + in the reverse transform. In Figure 2 we define the intervals:

- ΔT = time interval corresponding to ½ rotation of *clock 0* = $t_1 - t_0$
- ΔL = length of measuring rod $x_1 - x_0$
- ΔS = simultaneity adjustment $t_{clock\,1} - t_{clock\,0}$ where *clock 1* is the leading clock as seen from K’

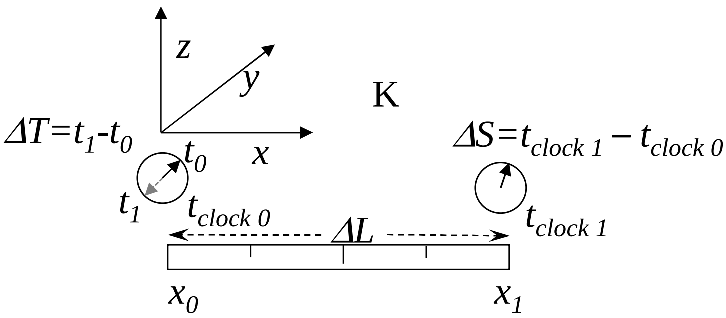

Fig. 2 – Intervals of time T, length L, and simultaneity S

In system K we assume that clocks have been synchronized by the Einstein synchronization method so that $t_{clock\,1} - t_{clock\,0} = 0$. Now we compute the corresponding Δ intervals from the point of view of K’:

$$\Delta T' = t'_1 - t'_0 = \gamma(t_1 - vx_0/c^2) - \gamma(t_0 - vx_0/c^2) = \gamma(t_1 - t_0) = \gamma\Delta T \qquad (2)$$

$$\Delta S' = \gamma(t_{clock1} - v(x + \Delta L)/c^2) - \gamma(t_{clock0} - vx/c^2) = -\gamma v \Delta L/c^2 \qquad (3)$$

For convenience let $t_0 = t_0' = 0$ at $x_0 = x_0' = 0$. Then $\Delta L = x_1$ and $\Delta L' = x_1'$ and we use the reverse transform to relate ΔL and $\Delta L'$ at time 0 in K’:

$$\Delta L = \gamma(\Delta L' + vt_0') = \gamma\Delta L' \Rightarrow \Delta L' = \Delta L/\gamma \qquad (4)$$

Notice that *clock 1* is encountered before *clock 0* by any observer in K waiting for them to pass. It is the leading clock of the pair.

3. Using light clocks to understand “leading clocks lag” and wave aberration

We may also use light clocks as Einstein often did to understand that leading clocks lag and answer student questions about why this must be so. Consider the example in Figure 3. On one hand it is not obvious to students or to the public why all clocks (e.g. biological clocks) must agree with the light clock. On the other hand, one of the most striking results of Special Relativity (SR) is the equivalence of matter and energy, specifically the ability to convert between matter and electromagnetic energy, and in view of that it is at least unsurprising that all material clocks agree with light clocks.

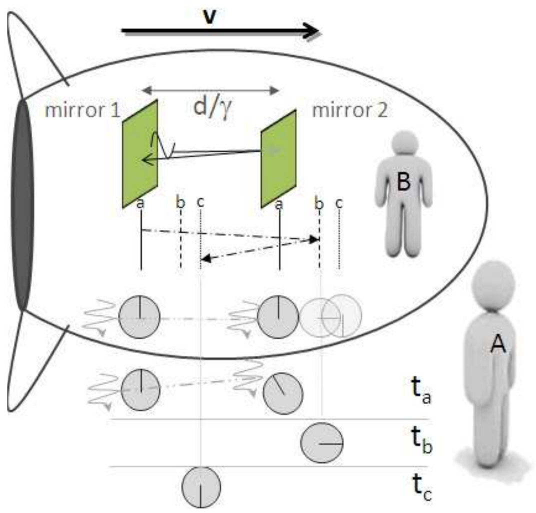

Fig. 3 - Using back and forth path of light to sync clocks implies *leading clocks lag*

The clocks drawn within the spaceship in Figure 3 are synchronized as viewed by observer B in the ship. Clocks drawn below the ship show observer A's view of the same clocks. Lines *a*, *b* and *c* show the position with respect to A of mirrors 1 and 2 at the time that a pulse of light strikes mirror 1, then mirror 2, and then mirror 1 again respectively. We may imagine there is also a sensor in the mirrors to compare pulse arrivals with the clocks, and some active optical system for regenerating energy lost from the pulse. B sees the clocks make 1/2 revolution for one round trip of a pulse. The clocks are shown pointing straight up at time t_a when a pulse leaves mirror 1. Ghost clocks on the right show clock readings of 1/4 for pulse arrival at mirror 2, and 1/2 for arrival back at mirror 1. No ghost clocks are shown for mirror 1 but they would be identical.

Observer A sees a shorter distance between the mirrors due to length contraction, but a longer overall light path because of the relative motion of B. Some older textbooks will show a transverse light clock and light path with a triangular derivation of the Lorentz gamma factor using the Pythagorean Theorem. In that case, the light path is longer by the factor γ. Since by assumption all clocks behave alike and the speed of light is constant in all directions, the length of the longitudinal light path in Figure 3 must also be longer by γ in A's view. Since clock intervals are also lengthened (dilated) by γ, the observer A easily explains how B arrives at the same value for the speed of light: light travels farther and B's clocks tick slower by the same factor.

Next we deal with how B and A describe the synchronization events. B will have the clocks transmit their readings to one another using the light pulse, and will set the clock at each mirror based on the transmitted value plus 1/2 the round trip light transit time. Thus by definition in B's view the transit times in each direction are equal. The following explains the time skew in a nutshell:

- For A the time for the light pulse to go from mirror 1 to 2 is longer than the return.
- In order for A and B to agree what clock 2 reads when the pulse arrives (and they must since it is a physical event at a point), the clock at mirror 2 must be *set back* or *lagged* as viewed by A in order to allow the extra time for transit from mirror 1 to 2.

Figure 3 also illustrates the connection of simultaneity and relativistic aberration. In the spaceship, two photons are shown as squiggly lines, with a point of constant phase diagrammed as a horizontal double-dotted line. In the view of the same two clocks seen by A below, at the lagging clock the point of constant phase hasn't arrived yet, so the wave front is tilted. If the speed of light were subject to vector addition as a normal velocity, then this wouldn't happen, but the relative speed of the wave would have to be different for each moving observer. Since each observer measures the same speed of light, then the direction of light must appear to change, which is relativistic aberration, addressed in section 7 of Einstein's 1905 paper on Special Relativity [1].

4. Applying all three memory aids

Because the memory aids are deltas between coordinate points they are not time dependent and are easier to use than coordinate transformations. But for the same reason it is easy to lose sight of the synchronized Einstein reference frame requirements for the memory aids to be valid.

Time Dilation

We must use two or more clocks in an Einstein synchronized reference frame to observe time dilation in a single moving clock under observation. Observations must be made only by reference frame clocks which are essentially adjacent to the clock under observation (i.e. as they pass). No remote observations are allowed. I call this *2-on-1* clock measurement, shown in Figure 4.

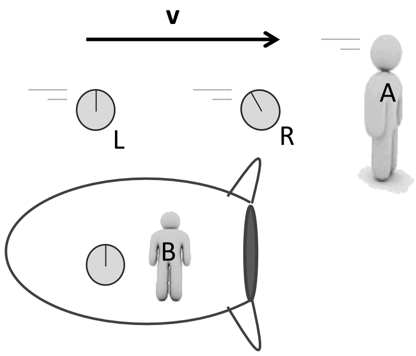

Fig. 4 – *2 on 1* clock measurement from viewpoint of observed clock B

We may of course make exceptions which are consistent with the spirit of this condition, such as using a single clock to observe particles moving in a circle in an accelerator which repeatedly pass the same reference frame clock, in which case it takes the role of two clocks. Strictly speaking the circular path is outside the domain of SR, but this technique works. We may also use signaling to make remote measurements provided we have compensated for all distances and delays as if the measurement were made only with local clocks.

In Figure 4, A has two synchronized clocks R and L. As B passes clock R, A records both clocks and labels the readings R and B_R. Similarly as B passes L, A records L and B_L. Since A is using an Einstein synchronized frame, we can infer that:

$$\begin{aligned} &\Delta T_B = \Delta T_A / \gamma \\ &\Leftrightarrow (L-R)/(B_L - B_R) = \gamma \end{aligned} \tag{5}$$

The observer B if using two clocks would reach the reciprocal conclusion, that time is dilated in A's frame, as it well known. But suppose B measures with just the one clock "B" against the two clocks R and L in passing? This measurement does not use an Einstein synchronized reference frame! The exact same times will be recorded as by A in (5), and B would conclude that time runs faster in A's frame. We may understand this as being similar to a traveler in an airplane crossing time zones. In B's frame, L and R are not synchronized. The leading clock, the one B encounters first, R, is lagging by some amount. As B passes clock L, the time difference includes not only the general rate of time passage in A's frame, but also adds the lag time that B sees between the two clocks R and L.

This *1 on 2* sort of observation is useful for determining what time it would be if B suddenly dropped into A's frame. It's used every day by airline passengers and schedulers, and could be used by the traveling twin in the traveling twins puzzle to determine the age of a stay-at-home twin. But it does not indicate the rate of ticking B sees in A's clocks when making a well-defined measurement using a reference frame with synchronized clocks. And one is only guaranteed of being able to work out all the laws of physics when using such a synchronized frame of reference.

Length Contraction

While length contraction would not seem to require clocks, if there is not a physical object involved we are indirectly compelled to use clocks to pin down the end points of the length involved. Otherwise students become confused about whether empty space contracts or not. A good rule of thumb is to ask if there was a material measuring stick spanning the length, which (if either) observer would see it as moving?

If we imagine a long measuring stick reaching from the surface of the earth to high in the atmosphere where mesons are formed from cosmic rays, then obviously the mesons would see it as moving, and thus contracted. So while we explain the ability of the mesons to reach the surface of the earth before they decay as being due to time dilation, a tiny alien traveling with the mesons would conclude that the distance to the surface of the earth is rather less than we claim.

If we instead imagine one end of the measuring stick held by (and traveling with) the tiny alien, then its other endpoint is not affixed to the surface of the earth, and we would argue that the leading edge of the measuring stick had a lagging clock as it passed, reading a much earlier time than what the alien claims is the instant of the creation of the high altitude mesons. Thus at the time claimed by the alien as "start," we conclude that in the alien's frame the leading endpoint has not reached earth's surface yet. Having a memory aid for "leading clocks lag" has allowed us nearly instant visualization of the problem.

Leading Clocks Lag

We will certainly observe that leading clocks lag when using an Einstein synchronized reference frame. But we do not have to use such a frame. The point observer B in Figure 3 observes that leading clocks lag in A's frame. Instead, the condition is that the *frame being observed* must be an Einstein synchronized reference frame, and that is not always true. In general it is not true for macroscopic (classical) clocks following finite acceleration, but it is true for coupled quantum systems and for classical systems that are accelerated infinitely slowly [4]. For other frames clock readings after acceleration will depend on how the acceleration is applied. It cannot be simultaneous in more than one frame, and it must be unequal at different points to achieve the correct length contraction after acceleration.

Application to de Broglie Waves

One of the most interesting applications of "leading clocks lag" is a simple derivation of de Broglie wavelength. De Broglie's original derivation [5] was based on the Lorentz transform, and our present version essentially follows this but with the use of our memory aids to simplify the derivation. The derivation is sufficiently obscure in history that it was recently recounted in a modern paper [6], and is completely omitted from many "explanations" of the "derivation" of de Broglie wavelength [7] [8]. This derivation also answers questions that have appeared on various physics student blogs asking how to perform the Lorentz transformation of de Broglie wavelength, which as it turns out is not the clearest way to ask the question [9].

De Broglie waves are based on Planck's idea of quantization of energy for photons, but applying the idea to matter by assuming matter has an associated "wave." De Broglie takes Planck's relations as given (we use f to distinguish frequency from velocity):

$$\begin{aligned} E &= hf \\ \lambda &= h / \rho \end{aligned} \tag{6}$$

The energy E is taken to be the relativistic energy $E=mc^2$. Wavelength is Planck's constant divided by momentum. For light, the speed c provided an obvious physical relationship between frequency and wavelength, but for matter the frequency is non-zero even when momentum is zero, and wavelength is infinite under those conditions. If velocity increases, frequency changes only slightly as it is dominated by the large rest mass energy, but wavelength rapidly decreases. This is not intuitive and bears to relation to light wavelengths, and is a source of confusion when performing coordinate transformations, because the de Broglie wavelength obviously cannot be handled as a real physical length.

Figure 5 shows de Broglie waves schematically in the rest frame of a particle. We should qualify this figure in several ways. Quantum particles do not strictly speaking have a known certain rest frame due to the uncertainty principle, but taking such complexity into consideration does not really add anything to the derivation of de Broglie wavelength. To the extent we approximately know the rest frame of a particle the de Broglie wavelength is approximately infinite. That is, the particle's associated wave vibrates, or you might think of it as "flashes," simultaneously throughout all space. A peak of the wave is like the tick of synchronized clocks. For illustration in the figure clocks at various points in space on a peak in time are shown as synchronized.

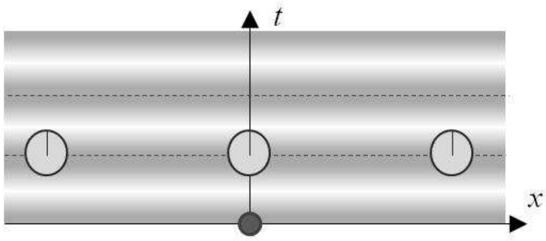

Fig. 5 – de Broglie waves in the rest frame of a particle

Note that de Broglie waves are not specifically shaped as probabilities like Schrodinger waves. De Broglie simply visualized a wave associated with a particle, and the probability interpretation came later. Also, de Broglie waves are based on the total energy of the particle including rest mass, while Schrodinger waves are based on kinetic energy. Since energy differences are all that matter physically, the two can be seen as differing only by a choice of measurement coordinates. It would appear that the frequencies are different, but again it is only differences that matter, and wavelengths, which are the property associated with physical manifestations like interference, are the same for de Broglie and Schrodinger waves (phase velocities, a calculated quantity not directly measurable, are different).

Notice that the de Broglie rest frame constitutes an Einstein synchronized reference frame. If we observe from another inertial frame using a point observer as shown in Figure 6 (we don't have any way to detect the wave peaks at other than a point using quantum measurement techniques), the wave peak clocks cannot all be synchronized, and thus the de Broglie wave cannot "flash" everywhere at once. As leading clocks will lag, the wave peak will occur a little later at leading points as viewed from a moving frame. Furthermore, since this is a *1 on 2* measurement process, the de Broglie frequency appears higher by a factor of γ rather than lower.

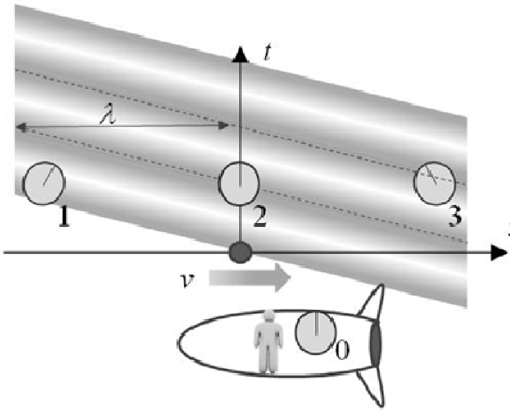

Fig. 6 – de Broglie waves seen from another inertial frame

If the de Broglie wave has frequency f, it will have period $1/f$, or in the observer's frame $1/\gamma f$. If we set the lagging time to the period and the wavelength λ to ΔL and solve for λ, we will arrive at the de Broglie wavelength:

$$1/\gamma f = \gamma v \Delta L / c^2 = \gamma v \lambda / c^2$$
$$\Leftrightarrow \lambda = c^2 / vf \quad (6)$$

Using $E=hf$ which alternately we can write as $f=E/h$ or $f=\gamma mc^2/h$, and $\rho=\gamma mv$ for relativistic momentum, we can obtain the more customary form:

$$\lambda = c^2 / v(\gamma mc^2 / h)$$
$$\Leftrightarrow \lambda = h / \gamma mv = h / \rho \quad (7)$$

As to the Lorentz transformation of λ, it is clear from the derivation that it does not have two physical endpoints in the rest frame of the particle but is already the product of a Lorentz transformation of simultaneity, and must be re-computed rather than illogically double transformed in order to get to yet another reference frame.

5. Conclusion

Three purely relative quantities have relations derived from the Lorentz transform that correspond to the memory aids "time dilation," length contraction," and "leading clocks lag." By understanding the constraints on reference frames (which ones are to be Einstein synchronized, and which may or must be point observers), students can quickly and reliably visualize and solve problems in Special Relativity involving multiple reference frames. Students will understand how quantum systems automatically constitute an Einstein synchronized frame, and how observing such systems at a single point as often required by quantum measurement can produce unexpected results.

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